Int. J Solids Structures, Vol. 11, pp. 275–294. Pergamon Press 1975. Printed in Great Britain

# TWO NONMIXED SYMMETRIC END-LOADINGS OF AN ELASTIC WAVEGUIDE

G. B. SINCLAIR
School of Engineering, University of Auckland, Private Bag, Auckland, New Zealand

and

JULIUS MIKLOWITZ
California Institute of Technology, Pasadena, California 91109, U.S.A.

(*Received* 22 *September* 1972; *revised* 1 *May* 1974)

**Abstract**—This investigation treats the response of the semi-infinite plate with free faces subjected to suddenly applied normal loads on its end. The plate is elastic and in plane strain. The normal loads are symmetric and act in the absence of shear stress, i.e. the plate has nonmixed end conditions. A double Laplace transform technique is used to obtain long-time information for two particular normal loads; the uniform load and the line-load. Near-field and far-field approximations are found. Results in the long-time near-field for the uniform load reduce to elementary forms; for the line-load however, the corresponding forms are quite complex entailing singular terms and some numerically evaluated contributions. The far-field approximations give rise to integral of the Airy function forms for both loads and, if the forces applied under the two loadings are equal, these far-field responses are shown to be identical.

## INTRODUCTION

Analysis of semi-infinite waveguides based on the equations of motion of linear elasticity is a subject of long-standing interest. Miklowitz[1] gives a review of the area up to 1969. As is pointed out in [1], contributions on *mixed* problems (combinations of stresses *and* displacements prescribed at the edge) are relatively numerous compared to those on the more physical *nonmixed* problems (stresses *or* displacements prescribed at the edge). This is because in the mixed problems a direct application of integral transforms leads to solutions whereas in the nonmixed problems it is impossible to find a combination of transforms which "asks for" the appropriate edge quantities. Benthem[2] developed a technique for handling this nonmixed character for the corresponding static problem in his paper on the stress analysis of elastostatic strips. In [1] this technique is extended to the elastodynamic case, the resulting method treating the semi-infinite waveguide in plane strain and composed of a homogeneous and isotropic linear material.

In the method Laplace transforms on time $t$ and the propagation coordinate $x$† are applied to the displacement equations of motion. For the plate with free faces,‡ expressions for the transformed displacements, $\tilde{\bar{u}}(s, y, p)$ and $\tilde{\bar{v}}(s, y, p)$,§ are subsequently obtained. These expressions contain all six *edge unknowns*: $\bar{u}(0, y, p)$, $\bar{v}(0, y, p)$, $(\partial\bar{u}/\partial y)(0, y, p)$, $(\partial\bar{v}/\partial y)(0, y, p)$, $(\partial\bar{u}/\partial x)(0, y, p)$ and $(\partial\bar{v}/\partial x)(0, y, p)$. Differentiation with respect to $y$ and application of the *edge*

†Within the plate $0 < x < \infty$ with $x = 0$ at the plate edge (or end).
‡The faces occur at $y = \pm h$, $y$ being the thickness coordinate of the plate.
§Here the Laplace transform on time $t$ is denoted by a bar and has parameter $p$ while the transform on $x$ is denoted by a tilde and has parameter $s$.

*conditions*† effectively reduces the number of edge unknowns to two. Now boundedness of the solution for $x \to \infty$ insists that $\bar{\bar{u}}(s, y, p)$ and $\bar{\bar{v}}(s, y, p)$ have no poles in the right-half $s$-plane. But $\bar{\bar{u}}(s, y, p)$ and $\bar{\bar{v}}(s, y, p)$ are found to have $R(s, p)$ in their denominators. And $R(s, p)$ set equal to zero is a generalized form of the *Rayleigh–Lamb frequency equation* for symmetric waves in a plate, which implies, for any given $p$ there exists an infinite set of points in the right-half $s$-plane at say, $s = s_j(p)$, such that $R(s_j(p), p) = 0$. Consequently the numerators of $\bar{\bar{u}}(s, y, p)$ and $\bar{\bar{v}}(s, y, p)$ must be set equal to zero at $s = s_j(p)$. When this is done the *boundedness condition* so furnished takes the form of a pair of coupled integral equations in the edge unknowns.‡ Solution of this boundedness condition then completes the determination of the edge unknowns.

To tackle the boundedness condition in [1], Miklowitz considers the case $p \to 0$ corresponding to the long-time. Since the *long-time near-field* in his problem involving a buiilt-in edge may reasonably be expected to be *static in nature*, this small-$p$ approximation enables him to follow the technique used by Benthem and represent the edge unknowns by *Fourier series* in the thickness coordinate $y$, in conjunction with the appropriate *singular terms* in $y$ drawn from *elastostatic theory*.§ These forms differ from Benthem's in that they are $p$-dependent; this dependence being present in the Fourier series coefficients $a_n(p)$ and $b_n(p)$, and in the singular term coefficient $a_0(p)$. Substituting these forms for the edge unknowns in the boundedness condition and integrating gives a pair of algebraic equations in the coefficients $a_0(p)$, $a_n(p)$ and $b_n(p)$ which hold for all $s = s_j(p)$. Carrying out the small-$p$ approximation for this pair of equations show the coefficients all to be ord$(1/p)$ (i.e. static in nature) as expected. Further for $p \to 0$, $s_j(p) = s_j$ the roots of

$$\sin 2sh + 2sh = 0\P$$

which lie in the first quadrant. These $s_j$ are the same values that occur in Benthem's analysis and, as in [2], Miklowitz then proceeds to a solution for the coefficients via the *method of reduction*. The attendent formal solution is then inverted using known techniques to provide long-time information for a nonmixed problem of the *displacement* type.

In the present work the method is employed to obtain long-time solutions to two nonmixed problems of the *stress* type. Both problems involve the longitudinal response of the plate to normal loads which are suddenly applied to the plate edge and act in the absence of shear stress. They are: Problem A, a *uniform normal load*; and Problem B, a *normal line-load* applied at the center of the plate edge. Problems A and B can be regarded as the first and limiting members, respectively, of the sequence of uniform normal loads acting on the interval $(-\Delta, \Delta)$ as $\Delta$ decreases from the plate half-thickness $h$, to zero. It follows that the responses found for the two problems can be expected to encompass the long-time behavior of the plate (with free faces) when it is suddenly subjected to *any* symmetric normal edge-loading.

In using the boundedness condition approach to derive the long-time responses for Problems A and B, the crux of the derivation lies in the choice of appropriate representations of the edge unknowns. These representations are constructed by adjoining the singular-term-plus-Fourier-series type representations of [1, 2] with forms which correspond to (in the transformed $p$-plane) the long-time time-dependence of the edge quantities. Such additional forms are obtained from

†The pair of conditions which prescribe either the stresses or the displacements at the plate edge.
‡*Cf.* the algebraic system found by Benthem in [2] for the static case.
§As Benthem states, the addition of such singular terms is necessary to ensure the convergence of the Fourier series.
¶Here $h$ is the plate half-thickness.

the associated *elementary theory*. Then, following a similar method to that of [1], the edge unknowns are determined in detail, leading to the formal solutions of the two problems. Inversion is then undertaken using known procedures.

Results in the *long-time near-field* for Problem A reduce to simple closed forms, the Fourier series coefficients being found to be zero in this instance. For Problem B the corresponding results involve time-dependent components, singular terms (of static character), and Fourier series with numerically evaluated coefficients. The *far-field long-time* approximations for the two problems lead to *integrals of the Airy function*—the same form that arises in mixed problems [1]. Further, comparison of these two far-field responses shows that if the *same force* is applied in both problems, the *same far-field response* will result.

## 1. GENERAL FORMULATION AND FORMAL SOLUTION. BOUNDEDNESS CONDITION

We now formulate the class of elastic waveguide problems to be considered. Plane rectangular cartesian coordinates are chosen for the plate section such that the $x$-axis is in the direction of propagation whilst the $y$-axis is in the thickness direction (Fig. 1). The displacement in the $x$-direction is $u$; in the $y$-direction, $v$. The plate thickness is $2h$.

The plate is composed of a homogeneous and isotropic linear elastic material. Under the assumptions of plane strain† the governing *displacement equations of motion* are

$$\begin{aligned} c_d^2\frac{\partial^2 u}{\partial x^2}(x,y,t)+(c_d^2-c_s^2)\frac{\partial^2 v}{\partial x\partial y}(x,y,t)+c_s^2\frac{\partial^2 u}{\partial y^2}(x,y,t)&=\frac{\partial^2 u}{\partial t^2}(x,y,t),\\ c_s^2\frac{\partial^2 v}{\partial x^2}(x,y,t)+(c_d^2-c_s^2)\frac{\partial^2 u}{\partial x\partial y}(x,y,t)+c_d^2\frac{\partial^2 v}{\partial y^2}(x,y,t)&=\frac{\partial^2 v}{\partial t^2}(x,y,t), \end{aligned} \tag{1.1}$$

on $\mathring{D}$, the interior of the domain $D=\{x,y|0\le x,-h\le y\le h\}$, and for $t>0$. Here $c_d=\sqrt{(\lambda+2\mu)/\rho}$ and $c_s=\sqrt{\mu/\rho}$ are the dilatational and equivoluminal body wave speeds respectively, with $\lambda$ and $\mu$ the Lamé constants and $\rho$ the mass density. The associated

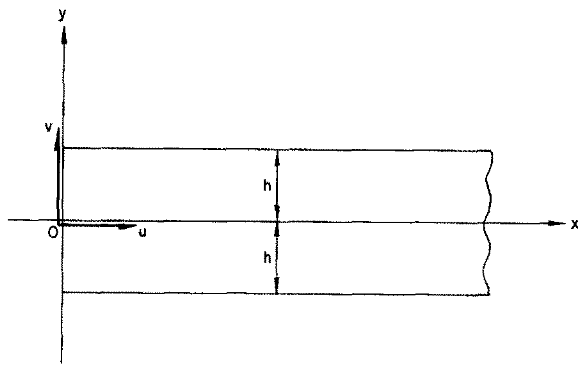


Fig. 1. Coordinates and displacements for the semi-infinite plate.

†If the direction perpendicular to the $xy$-plane is designated as the $z$-axis with $w$ the displacement in that direction then these plane strain assumptions require that $\partial/\partial z$ be a null operator and $w=0$.

*stress–displacement relations* are

$$\sigma_x(x,y,t)=\mu\left[k^2\frac{\partial u}{\partial x}(x,y,t)+(k^2-2)\frac{\partial v}{\partial y}(x,y,t)\right],$$
$$\sigma_y(x,y,t)=\mu\left[(k^2-2)\frac{\partial u}{\partial x}(x,y,t)+k^2\frac{\partial v}{\partial y}(x,y,t)\right], \tag{1.2}$$
$$\tau_{xy}(x,y,t)=\mu\left[\frac{\partial u}{\partial y}(x,y,t)+\frac{\partial v}{\partial x}(x,y,t)\right],$$

on $\mathring{D}$ for $t>0$. Here $\sigma_x$, $\sigma_y$, $\tau_{xy}$ denote the rectangular components of stress and $k^2=c_d^2/c_s^2$.

The plate is quiescent. Hence the *initial conditions* are

$$u(x,y,0)=v(x,y,0)=0,\quad \frac{\partial u}{\partial t}(x,y,0)=\frac{\partial v}{\partial t}(x,y,0)=0, \tag{1.3}$$

on $\mathring{D}$. *Conditions at infinity* are taken as

$$\lim_{x\to\infty}\left\{\begin{matrix} u(x,y,t),\dfrac{\partial u}{\partial x}(x,y,t),\ \text{etc.} \\ v(x,y,t),\dfrac{\partial v}{\partial x}(x,y,t),\ \text{etc.}\end{matrix}\right\}=0, \tag{1.4}$$

on $D$ for $t\geqslant 0$. Since the plate has stress-free surfaces at $y=\pm h$ we have the *plate-face conditions*

$$\sigma_y(x,h,t)=\tau_{xy}(x,h,t)=0, \tag{1.5}$$

for $x>0$, $t\geq 0$. Here we have suppressed the condition at $y=-h$ because we are only concerned with excitation of the plate which is symmetric with respect to the $x$-axis and for such excitation we have the *symmetry criteria*

$$v(x,0,t)=0,\qquad \tau_{xy}(x,0,t)=0, \tag{1.6}$$

for $x>0$, $t\geqslant 0$.† Finally we have our two *edge conditions*. The first of these specifies the normal stress which is suddenly applied at time $t=0$ on the edge, and the second maintains zero shear stress there. Equations (1.1)–(1.6) in conjunction with the edge conditions define the class of problems from which the two particular problems of ensuing sections will be drawn.

We now focus on a formal solution for this class of problems defining the Laplace transforms on $x$ (parameter $s$) and $t$ (parameter $p$) by

$$\tilde{f}(s)=\int_0^\infty f(x)\,\mathrm{e}^{-sx}\,\mathrm{d}x,\quad \bar{f}(p)=\int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t. \tag{1.7}$$

Double transformation of (1.1), followed by the application of (1.2), (1.3), (1.5), (1.6) and the

†As we have now restricted our investigation to symmetric loadings we need only consider the upper half-plate ($0\leq x$, $0\leqslant y\leq h$) from hereon.

attendant inversion formulae for (1.7), produces the forms that follow in equations (1.8)–(1.14).† First, the pertinent inversion integrals,

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \end{Bmatrix} = \frac{1}{2\pi i}\int_{\mathrm{Br}_p} \mathrm{e}^{pt}\left[\frac{1}{2\pi i}\int_{\mathrm{Br}_s} \mathrm{e}^{sx}\begin{Bmatrix} \bar{\tilde{u}}(s,y,p) \\ \bar{\tilde{v}}(s,y,p) \end{Bmatrix} \mathrm{d}s\right]\mathrm{d}p, \tag{1.8}$$

in which $\mathrm{Br}_p$ is the Bromwich contour in the $p$-plane, $\mathrm{Br}_s$ that in the $s$-plane. Here

$$\begin{aligned} \bar{\tilde{u}}(s,y,p) &= \bar{\tilde{u}}_1(s,y,p) + \bar{\tilde{u}}_2(s,y,p), \\ \bar{\tilde{v}}(s,y,p) &= \bar{\tilde{v}}_1(s,y,p) + \bar{\tilde{v}}_2(s,y,p), \end{aligned} \tag{1.9}$$

with

$$\begin{aligned} \bar{\tilde{u}}_1(s,y,p) &= \frac{1}{k_s^2}\int_0^y \left\{\left[\frac{s^2}{\alpha}\sinh\alpha(y-y') + \beta\sinh\beta(y-y')\right] f_1(s,y',p)\right. \\ &\qquad \left. + s[\cosh\alpha(y-y') - \cosh\beta(y-y')] f_2(s,y',p)\right\}\mathrm{d}y', \\ \bar{\tilde{v}}_1(s,y,p) &= \frac{1}{k_s^2}\int_0^y \left\{\left[\alpha\sinh\alpha(y-y') + \frac{s^2}{\beta}\sinh\beta(y-y')\right] f_2(s,y',p)\right. \\ &\qquad \left. + s[\cosh\alpha(y-y') - \cosh\beta(y-y')] f_1(s,y',p)\right\}\mathrm{d}y', \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \bar{\tilde{u}}_2(s,y,p) &= [C_1(s,p)\cosh\alpha y + C_2(s,p)\cosh\beta y]/R(s,p), \\ \bar{\tilde{v}}_2(s,y,p) &= \left[\frac{\alpha}{s}C_1(s,p)\sinh\alpha y + \frac{s}{\beta}C_2(s,p)\sinh\beta y\right]/R(s,p), \end{aligned} \tag{1.11}$$

where

$$R(s,p) = \gamma^2\cosh\alpha h\sinh\beta h + 4s^2\alpha\beta\sinh\alpha h\cosh\beta h, \tag{1.12}$$

and

$$\begin{aligned} C_1(s,p) &= -s[\gamma I_1(s,p)\sinh\beta h + 2s\beta I_2(s,p)\cosh\beta h], \\ C_2(s,p) &= \beta[2s\alpha I_1(s,p)\sinh\alpha h - \gamma I_2(s,p)\cosh\alpha h], \\ I_1(s,p) &= \frac{1}{k_s^2}\int_0^h \left\{\frac{s}{\alpha}[\gamma\sinh\alpha(h-y) + 2\alpha\beta\sinh\beta(h-y)] f_1(s,y,p)\right. \\ &\qquad \left. + [\gamma\cosh\alpha(h-y) - 2s^2\cosh\beta(h-y)] f_2(s,y,p)\right\}\mathrm{d}y + (k^2-2)\bar{\tilde{u}}(0,h,p), \\ I_2(s,p) &= \frac{1}{k_s^2}\int_0^h \left\{[2s^2\cosh\alpha(h-y) - \gamma\cosh\beta(h-y)] f_1(s,y,p)\right. \\ &\qquad \left. + \frac{s}{\beta}[2\alpha\beta\sinh\alpha(h-y) + \gamma\sinh\beta(h-y)] f_2(s,y,p)\right\}\mathrm{d}y - \bar{\tilde{v}}(0,h,p), \end{aligned} \tag{1.13}$$

†See Miklowitz [3] for details of the derivation of this set.

with

$$
\begin{aligned}
f_1(s, y, p) &= k^2\left[s\bar{u}(0, y, p)+\frac{\partial \bar{u}}{\partial x}(0, y, p)\right]+(k^2-1)\frac{\partial \bar{v}}{\partial y}(0, y, p),\\
f_2(s, y, p) &= s\bar{v}(0, y, p)+\frac{\partial \bar{v}}{\partial x}(0, y, p)+(k^2-1)\frac{\partial \bar{u}}{\partial y}(0, y, p).
\end{aligned}
\tag{1.14}
$$

Here $\alpha^2 = k_d^2 - s^2$, $\beta^2 = k_s^2 - s^2$, $\gamma = 2s^2 - k_s^2$, $k_d^2 = p^2/c_d^2$ and $k_s^2 = p^2/c_s^2$.

Observe that $\bar{\bar{u}}(s, y, p)$ and $\bar{\bar{v}}(s, y, p)$ involve, via $f_1(s, y, p)$ and $f_2(s, y, p)$ of (1.14), the six *edge unknowns*: $\bar{u}(0, y, p)$, $\bar{v}(0, y, p)$, $(\partial\bar{u}/\partial y)(0, y, p)$, $(\partial\bar{v}/\partial y)(0, y, p)$, $(\partial\bar{u}/\partial x)(0, y, p)$ and $(\partial\bar{v}/\partial x)(0, y, p)$. Differentiation with respect to $y$ of the first pair of these quantities yields the second pair. Application of the edge conditions further reduces the number of edge unknowns requiring determination to two. It remains for us to utilize the only condition left in the preceding formulation, to wit the infinity condition (1.4), to obtain a means for ascertaining these last two edge unknowns.

Note that the denominator of $\bar{\bar{u}}_2(s, y, p)$ and $\bar{\bar{v}}_2(s, y, p)$ given in (1.11) is $R(s, p)$ of (1.12). Now $R(s, p)$ set equal to zero is a generalized form of the *Rayleigh–Lamb frequency equation* for symmetric waves in an infinite elastic plate. Hence $R(s, p)$ has an infinite set of zeros in each quadrant of the complex $s$-plane for any given $p$-value. In particular there exists $s = s_j(p)$ such that

$$
R(s_j(p), p) = 0, \quad Re\, s_j(p) > 0. \tag{1.15}
$$

Accordingly, for $s = s_j(p)$ satisfying (1.15) and $C_1(s_j(p), p)$, $C_2(s_j(p), p)$ of (1.11) not equal to zero, $\bar{\bar{u}}(s, y, p)$ and $\bar{\bar{v}}(s, y, p)$ would have poles in the right-half $s$-plane. For such poles a residue evaluation of the inner integral in (1.8) would lead to exponentially unbounded waves as $x \to \infty$ – a violation of (1.4). Thus to eliminate the possibility of such transcendentally large terms $C_1(s_j(p), p)$ and $C_2(s_j(p), p)$ are set equal to zero. This yields *for* $s = s_j(p)$

$$
\begin{aligned}
&\frac{1}{k_s^2}\int_0^h \left\{\left[2s^2\frac{\cosh \alpha y}{\cosh \alpha h} - \gamma\frac{\cosh \beta y}{\cosh \beta h}\right]f_1(s, y, p) - \frac{s}{\beta}\left[2\alpha\beta\frac{\sinh \alpha y}{\cosh \alpha h}\right.\right.\\
&\qquad \left.\left. + \gamma\frac{\sinh \beta y}{\cosh \beta h}\right]f_2(s, y, p)\right\} \mathrm{d}y - \bar{v}(0, h, p) - (k^2-2)\psi\bar{u}(0, h, p) = 0,
\end{aligned}
\tag{1.16}
$$

with $\psi = (2s\alpha/\gamma)\tanh \alpha h = -(\gamma/2s\beta)\tanh \beta h$. Equation (1.16), the *boundedness condition*, corresponds to equation (23) of [1]. Its solution completes the determination of the edge unknowns.†

## 2. PROBLEM A: THE UNIFORM LOAD

In this section we use the boundedness condition of Section 1 to find the formal solution valid for small $p$—thence the long-time solution—for the specific problem of a uniform normal stress suddenly applied to the end of our waveguide (Fig. 2). Thus $\sigma_A$ is a positive quantity with the dimensions of stress; $U(t)$ is the *unit step function.*

†Observe that (1.16) is a complex-valued equation and thus furnishes a *pair* of coupled integral equations.

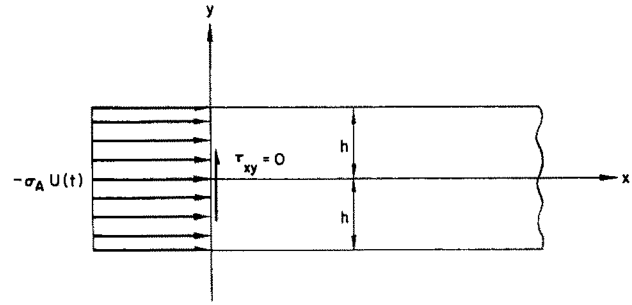


Fig. 2. Edge conditions for Problem A.

This particular loading is expressed through the stress–displacement relations (1.2) by

$$\bar{\sigma}_x(0,y,p)=\mu\left[k^2\frac{\partial\bar{u}}{\partial x}(0,y,p)+(k^2-2)\frac{\partial\bar{v}}{\partial y}(0,y,p)\right]=-\frac{\sigma_A}{p},$$
$$\bar{\tau}_{xy}(0,y,p)=\mu\left[\frac{\partial\bar{u}}{\partial y}(0,y,p)+\frac{\partial\bar{v}}{\partial x}(0,y,p)\right]=0, \tag{2.1}$$

for $0\le y\le h$, $Re\ p>0$. Here the quantities concerned have been transformed with respect to time so as to involve the edge unknowns explicitly.

The long-time behavior of the waveguides we are currently studying can be determined on inversion of the $\bar{\bar{u}}(s,y,p)$, $\bar{\bar{v}}(s,y,p)$ forms found for small $p$. It follows that the small-$p$ determination, via the boundedness condition, of the edge unknowns satisfying (2.1) will lead to the formal solution of the present problem. To accomplish this we must find suitable forms for the edge unknowns for small-$p$ which we can then substitute into (1.16) and thus "open up" the boundedness condition. In estalishing such forms for the edge unknowns we first turn to the *elementary theory* of compressional waves in a plate as our *modus operandi* for estimating those components of the edge unknowns which correspond to the long-time time-dependence of the displacements and displacement gradients at the edge: it being our hope that the introduction of these elementary theory terms will cause any remaining $y$-dependent contributions to the small-$p$ edge unknowns to be $O(1/p)$ (i.e. corresponding to terms which are either static in nature or decaying with time) and thus enable us to proceed in the manner of [1].

For the *elementary theory* the *equation of motion* is derived directly on assuming that the only stress component present is a normal stress which is uniform over any cross-section of the plate that is perpendicular to the direction of propagation of the long waves (the $x$-direction). Setting the second of (1.2) equal to zero and substituting the result found thereon for $(\partial v/\partial y)$ in the first of (1.2) furnishes the associated *constitutive relations*. On combining these expressions we have the standard field equations of the elementary theory,

$$\frac{\partial^2 u^e}{\partial x^2}(x,t)=\frac{1}{c_p^2}\frac{\partial^2 u^e}{\partial t^2}(x,t),$$
$$\frac{\partial v^e}{\partial y}(x,y,t)=-\left(\frac{k^2-2}{k^2}\right)\frac{\partial u^e}{\partial x}(x,t), \tag{2.2}$$
$$\sigma_x^e(x,t)=4\mu\left(\frac{k^2-1}{k^2}\right)\frac{\partial u^e}{\partial x}(x,t),$$

holding on $\mathring{D}$ for $t > 0$, where $c_p = c_s\sqrt{4(k^2-1)k^2}$ is the plate wave speed and the superscript $e$ intimates that the quantities involved are drawn from the elementary theory.†

To pose the analogous problem to Problem A for the elementary theory, we take a plate of unit width at rest, impinge on its end a force of magnitude $2\sigma_A h$ in the positive $x$-direction and thereafter retain only outward traveling waves (those propagating in the positive $x$-direction). For a plate subjected to these conditions, application of the Laplace transform on time readily produces, at the edge $x = 0$,

$$\bar{u}^e(0,p) = \frac{\hat{\sigma}_A}{pk_p}, \quad \frac{\partial \bar{u}^e}{\partial x}(0,p) = -\left(\frac{k^2}{k^2-2}\right)\frac{\partial \bar{v}^e}{\partial y}(0,y,p) = -\frac{\hat{\sigma}_A}{p}, \tag{2.3}$$

for $0 \le y \le h$, $Re\, p > 0$. Here $k_p = p/c_p$ and $\hat{\sigma}_A$ is the *dimensionless stress input for Problem A* defined by

$$\hat{\sigma}_A = \frac{\sigma_A}{4\mu}\left(\frac{k^2}{k^2-1}\right). \tag{2.4}$$

To the estimates in (2.3) we adjoin a supplementary set of edge unknowns—distinguished by the superscript $a$—to account for any differences between the elementary and exact theories.‡ For these additional contributions we employ the technique used in [1] and represent them by Fourier series in $y$ with the $p$-dependence incorporated into the series coefficients.

The *quarter-range Fourier series*§ chosen are

$$\begin{aligned} \frac{\partial \bar{u}^a}{\partial y}(0,y,p) &= \sum_{n=1,3,5,\ldots}^{\infty} A_n(p)\sin\frac{n\pi y}{2h}, \\ \frac{\partial \bar{v}^a}{\partial y}(0,y,p) &= \sum_{n=1,3,5,\ldots}^{\infty} B_n(p)\cos\frac{n\pi y}{2h}, \end{aligned} \tag{2.5}$$

for $0 \le y \le h$, $Re\, p > 0$. These series constitute a good choice in as much as they preclude the possibility of Gibbs' phenomena at the end points of the interval of representation $[0, h]$, which in turn ensures that the coefficients $A_n(p)$ and $B_n(p)$ must "go down" faster than $1/n$ as $n \to \infty$.¶ Such a decay will be of value in any subsequent numerical calculations.

Integrating (2.5) on invoking the first of the symmetry criteria (1.6) generates

$$\begin{aligned} \bar{u}^a(0,y,p) &= \bar{u}^c(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p)\cos\frac{n\pi y}{2h}, \\ \bar{v}^a(0,y,p) &= \sum_{n=1,3,5,\ldots}^{\infty} b_n(p)\sin\frac{n\pi y}{2h}, \end{aligned} \tag{2.6}$$

†It should be noted that (2.2) cannot in general be realized as a consistent specialization of the equations of motion and the stress–displacement relations quoted in Section 1.

‡By the term "exact theory" we will mean the plane strain simplification of the 3-dimensional theory of linear elasticity.

§It may easily be seen that the series representations in (2.5) can be drawn from the half-range series on $[0, 2h]$ in much the same manner that the half-range series on $[0, h]$ can be extracted from the full Fourier series on $[-h, h]$. Thus a Fourier theorem holds true for the series of (2.5) and the name "quarter-range Fourier series" is appropriate.

¶The proof of this result rests largely on the Riemann–Lebesque lemma—see, for example [4].

for $0 \le y \le h$, $Re\ p > 0$. Here we have introduced for the transformed *supplemental corner displacement* in the $x$-direction $\bar{u}^a(0, h, p)$ the notation $\bar{u}^c(p)$ and exchanged $(-2h/n\pi)A_n(p)$ for $a_n(p)$, $(2h/n\pi)B_n(p)$ for $b_n(p)$. Observe that our method of procuring (2.6) establishes the validity of term-by-term differentiation of the series there. Notice also that as a consequence of the large-$n$ behavior secured for $A_n(p)$, $B_n(p)$, the coefficients in (2.6) must obey the order conditions

$$a_n(p) = o\left(\frac{1}{n^2}\right), \quad b_n(p) = o\left(\frac{1}{n^2}\right), \quad \text{as} \quad n \to \infty. \tag{2.7}$$

We now combine these additional terms with those from the elementary theory. To do this we integrate the last of (2.3) to provide a "$\bar{v}^e(0, y, p)$" (once again invoking the first of (1.6)) and take $(\partial \bar{u}^a/\partial x)$ and $(\partial \bar{v}^a/\partial x)$ so that when added to the corresponding elementary terms, satisfaction of the edge conditions (2.1) is assured. We thus obtain

$$\begin{aligned}
\bar{u}(0, y, p) &= \frac{\hat{\sigma}_A}{pk_p} + \bar{u}^c(p) + \sum_{n=1,3,5,\dots}^{\infty} a_n(p) \cos\frac{n\pi y}{2h}, \\
\bar{v}(0, y, p) &= \left(\frac{k^2-2}{k^2}\right)\frac{\hat{\sigma}_A}{p} y + \sum_{n=1,3,5,\dots}^{\infty} b_n(p) \sin\frac{n\pi y}{2h}, \\
\frac{\partial \bar{u}}{\partial x}(0, y, p) &= -\frac{\hat{\sigma}_A}{p} - \left(\frac{k^2-2}{k^2}\right) \sum_{n=1,3,5,\dots}^{\infty} \frac{n\pi}{2h} b_n(p) \cos\frac{n\pi y}{2h}, \\
\frac{\partial \bar{v}}{\partial x}(0, y, p) &= \sum_{n=1,3,5,\dots}^{\infty} \frac{n\pi}{2h} a_n(p) \sin\frac{n\pi y}{2h},
\end{aligned} \tag{2.8}$$

for $0 \le y \le h$, $Re\, p > 0$. Equation (2.8) concludes the postulation of the edge unknowns for Problem A since differentiation with respect to $y$ of the first two expressions therein gives $(\partial \bar{u}/\partial y)$ and $(\partial \bar{v}/\partial y)$.

We can now proceed to open up the boundedness condition (1.16). Substituting (2.8) into (1.16) (via (1.14)) and performing the subsequent simple integrations yields

$$\sum_{n=1,3,5,\dots}^{\infty} [C_a(n; s, p)a_n(p) + C_b(n; s, p)b_n(p)] + C_c(s, p)\bar{u}^c(p) = F_A(s, p), \tag{2.9}$$

for $s = s_j(p)$ (satisfying (1.15)), where

$$\begin{aligned}
C_a(n; s, p) &= (-)^{(n-1)/2} \frac{n\pi s}{h}\left(\frac{k^2-1}{k^2}\right)\left[\frac{n^2\pi^2}{2h^2} + k_s^2\right] \Big/ \alpha_n^2\beta_n^2, \\
C_b(n; s, p) &= (-)^{(n-1)/2}\left[\frac{n^2\pi^2 s^2}{h^2}\left(\frac{k^2-1}{k^2}\right) - \alpha_n^2 k_s^2\right] \Big/ \alpha_n^2\beta_n^2, \\
C_c(s, p) &= k_d^2(k^2-2)\psi/\alpha^2, \\
F_A(s, p) &= \frac{\hat{\sigma}_A k_d^2}{p\alpha^2\beta^2}(k^2-2)\left[\left(\frac{\beta^2}{k_p} - \frac{sk_s^2}{2k_p^2}\right)\psi + \alpha^2 h\right],
\end{aligned} \tag{2.10}$$

with $\alpha_n^2 = \alpha^2 + (n\pi/2h)^2$ and $\beta_n^2 = \beta^2 + (n\pi/2h)^2$. Equations (2.9) and (2.10) comprise an infinite set of linear equations for $\bar{u}^c(p)$ and the Fourier coefficients $a_n(p)$, $b_n(p)$ $(n = 1, 3, 5, \ldots)$. We now seek a solution of this set for small $p$, considering first the $s_j(p)$ as $p \to 0$.

Asymptotics on $R(s, p)$ of (1.12) give

$$R(s, p) = -is^2 k_d^2 (k^2 - 2) r(s) + O(p^4) \quad \text{as} \quad p \to 0, \tag{2.11}$$

where $r(s) = \sin 2sh + 2sh$. Hence $s_j = \lim_{p \to 0} s_j(p)$ $(j = 1, 2, \ldots)$ are selected from the complex zeros of $r(s)$ as those roots having a positive real part. Indeed only those roots in the first quadrant are required in the present analysis because the elements in (2.9) are symmetric about the real $s$-axis (i.e. $C_a^*(n; s, p) = C_a(n; s^*, p)$, $C_b^*(n; s, p) = C_b(n; s^*, p)$, $C_c^*(s, p) = C_c(s^*, p)$ and $F_A^*(s, p) = F_A(s^*, p)$ where the * denotes the complex conjugate of the quantity beneath it). Robbins and Smith[5] list (in order of increasing real part) the first ten values of $2sh$ satisfying $r(s) = 0$ in the first quadrant.

It is shown in [4] that

$$\lim_{p \to 0} \left\{ \frac{\mathrm{d}s_j}{\mathrm{d}p} \right\} = 0 \quad (j = 1, 2, \ldots).\dagger \tag{2.12}$$

It follows, as remarked in [1], that the zeros of $r(s)$ in (2.11) are a good approximation to the $s_j(p)$ for a range of $p$ small but greater than zero.

Continuing with our solution of (2.9) for small-$p$ we consider the behavior of the "unknowns" as $p \to 0$. On the basis of the premise that our elementary theory will in fact describe the nature of the dominant time variation in the very long-time, we require $\operatorname{ord}\{\bar{u}^c(p)\} \le \operatorname{ord}\{p^{-2}\}$, $\operatorname{ord}\{a_n(p)\} \le \operatorname{ord}\{p^{-1}\}$ and $\operatorname{ord}\{b_n(p)\} \le \operatorname{ord}\{p^{-1}\}$ for $p \to 0$. Moreover, for these terms to have significant contributions to the long-time solution we need the orders of all three quantities to be greater than one. *In toto* then we seek $\bar{u}^c(p)$, $a_n(p)$, $b_n(p)$ such that

$$\begin{aligned} &\operatorname{ord}\{1\} < \operatorname{ord}\{u^c(p)\} \le \operatorname{ord}\{p^{-2}\} \quad \text{as} \quad p \to 0, \\ &\operatorname{ord}\{1\} < \begin{Bmatrix} \operatorname{ord}\{a_n(p)\} \\ \operatorname{ord}\{b_n(p)\} \end{Bmatrix} \le \operatorname{ord}\{p^{-1}\} \quad \text{as} \quad p \to 0. \end{aligned} \tag{2.13}$$

Now expanding the terms in (2.10) for $p \to 0$ and substituting the resulting expressions into the set of equations for $\bar{u}^c(p)$, $a_n(p)$, $b_n(p)$—namely into (2.9)—produces, in view of the order requirements (2.13),

$$\sum_{n=1,3,5,\ldots}^{\infty} (-)^{(n+1)/2} \frac{n^2 s}{s_n^4} \left[ \frac{n\pi}{2h} a_n(p) + s b_n(p) \right] = 0 \quad \text{as} \quad p \to 0,$$

$$\text{for} \quad s = s_j (j = 1, 2, \ldots), \tag{2.14}$$

where $s_n^2 = s^2 - (n\pi/2h)^2$. Clearly (2.14) admits the solution $a_n(p) = b_n(p) = 0$ for $p \to 0$ $(n = 1, 3, 5, \ldots)$. This solution insists that, for the present small-$p$ approximation, any contributions to the edge unknowns other than those derived from the elementary theory must be

†Equation (2.12) corresponds to the result that the $\kappa_j$ branches, with the exception of the lowest mode $\kappa_0$, are normal to the plane $\omega = 0$ in the $\kappa$–$\omega$ frequency spectrum.

confined to $\bar{u}^c(p)$. Further, since the boundedness condition has been freed of $\bar{u}^c(p)$ in (2.14), $\bar{u}^c(p)$ endures as an unknown at this juncture. This indeterminancy may be attributed to the problem in the near-field long-time domain asymptotically approaching a second boundary-value problem in elastostatics (stresses prescribed), since this type of static problem admits an arbitrary rigid displacement field.

The elastodynamic problem of the second type however, has no such arbitrary displacement field which fact suggests that we turn to the far-field response in the present problem and look at the boundedness condition as $s$ and $p$ tend to zero together. Under this limiting procedure, $R(s,p)=0$ gives as the lowest mode

$$s_0 = \pm k_p + O(p^3) \quad \text{as} \quad p \to 0.\dagger \tag{2.15}$$

Taking the positive sign in (2.15) defines an $s_0$ which, for $Re\, p > 0$ and a range of $p$ small but not equal to zero, satisfies the requirements of (1.15). Hence the boundedness condition applies and we take $s = s_0$ and $p \to 0$ in (2.9), (2.10) to obtain, in view of the order requirements (2.13) and the solution of (2.14),

$$\bar{u}^c(p) = \sum_{n=1,3,5,\dots}^{\infty} (-)^{(n-1)/2} \frac{2}{n\pi} a_n(p) = 0 \quad \text{as} \quad p \to 0. \tag{2.16}$$

Equation (2.16) completes the determination of the small-$p$ edge unknowns for Problem A. We next focus our attention on the attendant formal solution.

Substituting the reduced version of (2.8) ($\bar{u}^c(p)=0$, $a_n(p)=b_n(p)=0$) into the general formal solution contained in equations (1.9) through (1.14) gives rise to

$$\begin{aligned}\bar{\bar{u}}(s,y,p) &= \frac{-\hat{\sigma}_A}{p\alpha^2\beta^2}\left[\frac{s}{k_p}(\beta^2+k_p s) - \frac{k_d^2}{k^2}(3k^2-2) + \frac{1}{R(s,p)}\{\Theta^u(s,y,p)+\Phi^u(s,y,p)\}\right],\\ \bar{\bar{v}}(s,y,p) &= \frac{-\hat{\sigma}_A}{p\alpha^2\beta^2}\left[\left(\frac{k^2-2}{k^2}\right)s\alpha^2 y + \frac{1}{R(s,p)}\left\{\frac{\alpha}{s}\Theta^v(s,y,p)+\frac{s}{\beta}\Phi^v(s,y,p)\right\}\right],\end{aligned} \tag{2.17}$$

with $\Theta^u = k_d^2(k^2-2)s(\beta^2/k_p)[\gamma \sinh \beta h \cosh \alpha y + 2\alpha\beta \sinh \alpha h \cosh \beta y]$ and similar expressions for $\Phi^u$, $\Theta^v$ and $\Phi^v$ which are suppressed here in the interests of brevity.‡ Equation (2.17) holds for small $p$. In tackling its inversion we treat three ranges of $s$ separately: $(s/p)\to\infty$ as $p\to 0$; $(s/p)=c$ as $p\to 0$, $c$ a nonzero constant; and $(s/p)\to 0$ as $p \to 0$.

For the first of these ranges—which corresponds to the *near-field*—(2.17) becomes

$$\begin{aligned}\bar{\bar{u}}(s,y,p) &= \frac{\hat{\sigma}_A}{ps}\left[\frac{c_p}{p}-\frac{1}{s}\right] + O(1) \quad \text{as} \quad p\to 0,\\ \bar{\bar{v}}(s,y,p) &= \frac{\hat{\sigma}_A}{ps} y + O(1) \quad \text{as} \quad p \to 0,\end{aligned} \tag{2.18}$$

†Setting $s_0 = i\kappa_0$ $p = i\omega$ in (2.15) gives $\kappa_0 = \omega/c_p$ as the lowest mode in the $\kappa$-$\omega$ frequency spectrum—the same expression as may be established for the elementary theory of (2.2). This is the reason underlying the close agreement of the exact and elementary theories for the long-time responses of waveguides to low frequency inputs.

‡See [4] for details.

for $0 \leq y \leq h$, $Re\, p > 0$. Inversion of (2.18) then gives

$$u(x, y, t) \sim \hat{\sigma}_A[c_p t - x] \quad \text{as} \quad t \to \infty,$$
$$v(x, y, t) \sim \left(\frac{k^2 - 2}{k^2}\right)\hat{\sigma}_A y \quad \text{as} \quad t \to \infty, \tag{2.19}$$

for $0 < x < X, 0 \leq y \leq h$, where $X$ demarks the extent of the near field. In view of the forms found for the edge unknowns (see (2.8) with $\bar{u}^c(p)$, $a_n(p)$, $b_n(p)$ set equal to zero therein) the region of validity for (2.19) may be extended to include $x = 0$. Further, due to the uniformity in $s$ and $y$ of the asymptotics on $p$ for this case, (2.19) may be differentiated with respect to $x$ and $y$ to produce the near-field long-time strains.

Turning to the case of $s$ and $p$ tending to zero concurrently—which corresponds to being in the vicinity of the *wavefront*—(2.17) subject to this limiting procedure gives

$$\bar{\bar{u}}(s, y, p) = \frac{\hat{\sigma}_A}{pk_p(s + k_p)}[1 + O(p^2)] \qquad \text{as} \quad s, p \to 0,$$
$$\bar{\bar{v}}(s, y, p) = \left(\frac{k^2 - 2}{k^2}\right)\frac{\hat{\sigma}_A y}{p(s + k_p)}[1 + O(p^2)] \quad \text{as} \quad s,p \to 0, \tag{2.20}$$

for $0 \leq y \leq h$, $Re\, p > 0$. Inversion of the terms in (2.20) then produces

$$u(x, y, t) \sim \hat{\sigma}_A[c_p t - x]U(c_p t - x) \qquad \text{as } t \to \infty,$$
$$v(x, y, t) \sim \left(\frac{k^2 - 2}{k^2}\right)\hat{\sigma}_A y U(c_p t - x) \qquad \text{as } t \to \infty. \tag{2.21}$$

Comparison of (2.21) with (2.19) demonstrates that the former applies on $D$. The analogous asymptotics on $s\bar{\bar{u}}$ and $(\partial\bar{\bar{v}}/\partial y)$ furnish

$$\frac{\partial u}{\partial x}(x, y, t) \sim -\left(\frac{k^2}{k^2 - 2}\right)\frac{\partial v}{\partial y}(x, y, t) \sim -\hat{\sigma}_A U(c_p t - x) \quad \text{as} \quad t \to \infty, \tag{2.22}$$

on $D$. Equations (2.21), (2.22) constitute our *first far-field approximation* for Problem A and exhibit waves that are non-decaying in space and time with (2.22) attesting to the long-time *Poisson's ratio coupling* of the *longitudinal* and *thickness strains* ($k^2$ may be expressed as a function of Poisson's ratio alone). Interesting is the fact that (2.21), (2.22) could have been obtained from the elementary theory directly, though this is *not* what we have done in the present treatment. Rather, we have shown that the elementary theory solution is obtained from the first long-time approximation of the exact theory for Problem A.

A higher order approximation than (2.22), based on a second approximation to $R(s, p)$, is available for both $s$ and $p$ tending to zero and has been derived in work on a related mixed problem for the circular bar (see Skalak[6]). This higher order approximation gives more of the wave features than (2.22) and closely represents the physical nature of a longitudinal pulse in the far-field of an end-loaded rod or plate. Accordingly this approximation will be derived here.

Taking $s, p \to 0$ in $R(s, p) = 0$ and keeping higher order terms gives, as the branch having

$Re\, s \leq 0$,

$$s_0 = -k_p\left(1 - \frac{\xi}{3}k_p^2\right) + O(p^5) \quad \text{as} \quad p \to 0,\dagger \tag{2.23}$$

where $\xi = (h^2/2)[(k^2-2)/k^2]^2$ and $Re\ p > 0$. Consequently, on considering the counterpart of (2.20) for $s\bar{\bar{u}}$ and $(\partial\bar{\bar{v}}/\partial y)$, one sees that the residues associated with the inner integral of the double inversion formula (1.8) at $s = s_0$ are given by

$$\begin{Bmatrix} \text{Residue} \\ \text{at} \\ s = s_0 \end{Bmatrix} \begin{Bmatrix} s\bar{\bar{u}}(s, y, p) \\ \dfrac{\partial\bar{\bar{v}}}{\partial y}(s, y, p) \end{Bmatrix} = -\frac{\hat{\sigma}_A}{p} e^{-k_p x[1-(\xi/3)k_p^2]} \begin{Bmatrix} 1 \\ -\left(\dfrac{k^2-2}{k^2}\right) \end{Bmatrix} + O(p), \tag{2.24}$$

for $p \to 0$, $Re\, p > 0$. Using (2.24) to evaluate the inner integral in (1.8) then produces

$$\frac{\partial u}{\partial x}(x, y, t) \sim -\left(\frac{k^2}{k^2-2}\right)\frac{\partial v}{\partial y}(x, y, t) \sim -\frac{\hat{\sigma}_A}{2\pi i}\int_{\mathrm{Br}_p} e^{p[t-(x/c_p)(1-[\xi/3]k_p^2)]}\frac{\mathrm{d}p}{p}, \tag{2.25}$$

for $p \to 0$ and $x, t$ large. The first of (2.25) reiterates the Poisson's ratio coupling of (2.22) for the present approximation, enabling us to focus our attention on $(\partial u/\partial x)$ alone from henceforth. Now the integrand in the expression for $(\partial u/\partial x)$ in (2.25) decays as $\mathrm{Im}\ p \to \infty$. Thus $\mathrm{Br}_p$ may be translated to the imaginary $p$-axis provided we indent the contour near the origin. Collapsing this indentation—thereby collecting the residue contribution from the simple pole at $p = 0$—setting $p = i\omega$ and combining the resulting integrals from $-\infty$ to 0 and from 0 to $\infty$ leads to

$$\frac{\partial u}{\partial x}(x, y, t)/\hat{\sigma}_A \sim \frac{1}{\pi}\int_0^\infty \sin\left(\frac{\omega x}{c_p}\left[1 + \frac{\xi}{3}\frac{\omega^2}{c_p^2}\right] - \omega t\right)\frac{\mathrm{d}\omega}{\omega} - \frac{1}{2}, \tag{2.26}$$

for $x, t \to \infty$. Using the integral representation of the Airy function‡ (2.26) may be reduced to

$$\frac{\partial u}{\partial x}(x, y, t) \sim -\hat{\sigma}_A\left[\int_0^\eta Ai(-\eta')\,\mathrm{d}\eta' + \frac{1}{3}\right], \tag{2.27}$$

for $x, t \to \infty$, where $Ai(-\eta')$ is the Airy function and $\eta = (c_p t - x)/\sqrt[3]{\xi x}$. Note that (2.27), our *second far-field approximation*, is also non-decaying in space and time but unlike our first approximation (see (2.22)) exhibits dispersion. To compare the two approximations Fig. 3 shows a plot of them both.

To conclude the inversion of the small-$p$ formal solution for Problem $A$ we consider the third $s$-range, namely $p$ small with $s \to 0$. Under such a limiting procedure, (2.17) establishes that $u$, $v$ are $O(1)$. Hence we have no contributions to the displacements for $t$ large, $x \to \infty$. Similarly it may be shown that the strains are zero as $x \to \infty$ and thus our solution is in accord with the conditions at infinity (1.4).

†*Cf.* equation (2.15).
‡See, for example, Abramowitz and Stegun[7], p. 447.

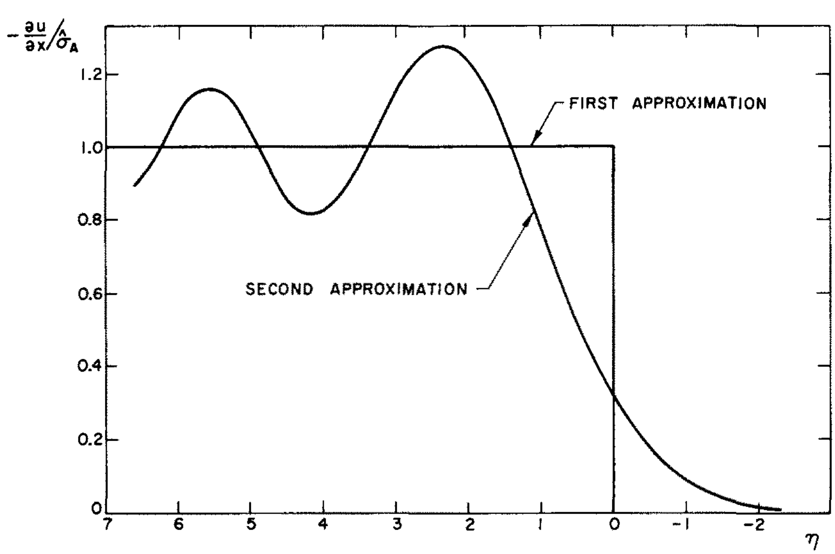


Fig. 3. Long-time far-field response of the longitudinal strain.

## 3. PROBLEM B: THE LINE-LOAD

Following the procedures adopted in Section 2 we now seek long-time information for a line-load impact on the end of our wave-guide (Fig. 4). Thus $\sigma_B$ is a positive quantity with the

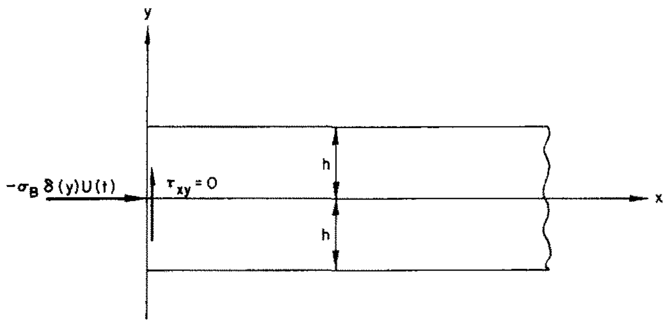


Fig. 4. Edge conditions for Problem B.

dimensions of force per unit length; $\delta(y)$ is the *symmetric delta*, a generalized function defined by

$$\delta(y)=\lim_{\Delta\to 0}\delta(y;\Delta),\quad \delta(y;\Delta)=\frac{1}{2\Delta}[U(y+\Delta)-U(y-\Delta)](\Delta>0). \tag{3.1}$$

Operations on $\delta(y)$ will be defined in general as the result of performing the equivalent operation on $\delta(y;\Delta)$ then taking the limit $\Delta\to 0$. In view of this, the edge conditions for Problem B—which may be expressed through (1.2) by

$$\begin{aligned}\bar{\sigma}_x(0,y,p)&=\mu\left[k^2\frac{\partial\bar{u}}{\partial x}(0,y,p)+(k^2-2)\frac{\partial\bar{v}}{\partial y}(0,y,p)\right]=-\frac{\sigma_B}{p}\delta(y),\\ \bar{\tau}_{xy}(0,y,p)&=\mu\left[\frac{\partial\bar{u}}{\partial y}(0,y,p)+\frac{\partial\bar{v}}{\partial x}(0,y,p)\right]=0,\end{aligned} \tag{3.2}$$

for $0 \le y \le h$, $Re\, p > 0$—can be regarded as the conditions satisfied by the problem arrived at as the limit, $\Delta \to 0$, of the sequence of normal loadings $\sigma_x(0, y, t) = -\sigma_B \delta(y; \Delta) U(t)$ acting on the edge of our plate.

As in Section 2 we must now postulate forms for the edge unknowns in order to open up the boundedness condition (1.16). In undertaking this postulation, the only differences encountered between the pattern established for Problem A and that required for Problem B will be occasioned by the singular nature of the latter problem. To facilitate the appraisal of such singular nature we resolve Problem B, for the long-time, into three problems (Fig. 5). The first of these, Problem B1, is the line-load on the elastostatic half-space, sometimes referred to as the Flamant problem. Problem B1 will provide the long-time *singular parts of the edge unknowns*. The second problem, Problem B2, is the residual associated elastostatic problem. The stresses applied on the plate-faces here ($\sigma_y'$ and $\pm\tau_{xy}'$) are of the same magnitude as those acting on the corresponding sections of the half-space in Problem B1 but opposite in sign. Problem B2 is made self-equilibrating by the introduction of a uniform normal stress, $\sigma_B/2h$, acting on the plate edge. The attendant edge values for this problem will contribute to the *regular parts of the edge unknowns*. The third problem, Problem B3, is the uniform normal load applied to our waveguide, recognizable as Problem A with a modified stress input. Problem B3 will furnish the dominant time-dependence of the edge quantities in the long-time, thereby completing the selection of the representations for the regular parts of the edge unknowns.

We now consider each problem in this decomposition individually; first, the singular problem. For the Flamant problem at $x = 0$ we have†

$$\begin{aligned}
\bar{u}^s(0, y, p) &= -\frac{\hat{\sigma}_B h}{p} \ln \frac{y}{h}, \quad \frac{\partial \bar{u}^s}{\partial y}(0, y, p) = -\frac{\partial \bar{v}^s}{\partial x}(0, y, p) = -\frac{\hat{\sigma}_B h}{py}, \\
\bar{v}^s(0, y, p) &= -\frac{\hat{\sigma}_B \pi h}{2pk^2} \operatorname{sgn}(y), \quad \frac{\partial \bar{v}^s}{\partial y}(0, y, p) = \frac{\partial \bar{u}^s}{\partial x}(0, y, p) = -\frac{\hat{\sigma}_B \pi h}{pk^2} \delta(y),
\end{aligned} \tag{3.3}$$

for $0 < y \le h$, $Re\, p > 0$—with extension to include $y = 0$ in an integrable sense whenever this is asked for by the boundedness condition, (1.16).‡ Here $\operatorname{sgn}(y) = 2U(y) - 1$ is the signum function ($\operatorname{sgn}(0) = 0$ by definition) and $\hat{\sigma}_B$ is the dimensionless stress input for Problem B defined by

$$\hat{\sigma}_B = \frac{\sigma_B}{2\mu\pi h} \frac{k^2}{k^2 - 1}. \tag{3.4}$$

Note that $\bar{u}^s(0, y, p)$ in (3.3) is determined to within an arbitrary rigid-body displacement; such indefiniteness is consistent with a second boundary-value problem in elastostatics and will be incorporated in a $\bar{u}^c(p)$ term in the same manner as in Section 2.

One should bear in mind that currently we are merely postulating the forms for the singular parts of the edge unknowns. Accordingly (3.3) is only a reasonable guess as to what these terms might be, based on the thesis that, in the near-field, the long-time singular nature of a problem involving, exclusively, outward propagating disturbances is the same as the singular behavior of the corresponding elastostatic problem. Should it transpire that this is not the case for Problem B, the convergence of the Fourier series that we subsequently select will, at best, be attained only in some generalized sense.

†See the Appendix at the end of the paper for a sketch proof of (3.3).

‡As intimated earlier, the integral from zero to some upper limit of an integrand containing $\delta(y)$ will be defined to be equal to the limit, $\Delta \to 0$, of the equivalent integral entailing $\delta(y; \Delta)$.

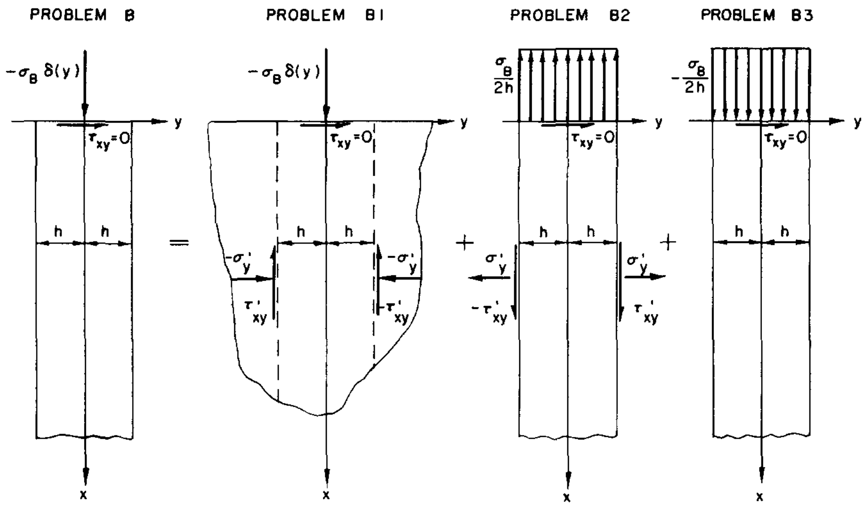


Fig. 5. Decomposition of Problem B.

For use in Problem B2 we set $y = h$ in the stress distribution associated with the Flamant problem to obtain

$$\tau'_{xy} = (x/h)\sigma'_y = \frac{2\sigma_B}{\pi h}\frac{(x/h)^2}{[1+(x/h)^2]^2}, \quad (x \geq 0). \tag{3.5}$$

In choosing edge forms for Problem B2 we observe the similarity of the prescribed stresses at $x = 0$ for this problem with those in Problem A. This similarity suggests that suitable representations for the edge unknowns in Problem B2 would be the expressions in (2.8) with $\hat{\sigma}_A$ therein replaced by $-\pi\hat{\sigma}_B/4$. However, Problem B2 has in addition to its edge stresses the equilibrating plate-face stresses $\sigma'_y$ and $\tau'_{xy}$.† This fact further suggests that we modify the forms arrived at subsequent to the replacement by dropping the $-\pi\hat{\sigma}_B/4pk_p$ term.

Now turning to Problem B3 we note its complete equivalence to Problem A subject to $\sigma_B/2h$ being equal to $\sigma_A$. It follows that the appropriate forms for Problem B3 are obtained from (2.8) on exchanging $\hat{\sigma}_A$ there for $\pi\hat{\sigma}_B/4$.

Combining the edge unknown representations for these last two problems will cancel the elements in $\bar{v}(0, y, p)$, $(\partial\bar{u}/\partial x)(0, y, p)$ which give rise to $\pm\pi\sigma_B/2h$ and thus produce the compact forms,

$$\begin{aligned}
\bar{u}^r(0, y, p) &= \frac{\hat{\sigma}_B\pi}{4pk_p} + \bar{u}^c(p) + \sum_{n=1,3,5,\ldots}^{\infty} a_n(p)\cos\frac{n\pi y}{2h}, \\
\bar{v}^r(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty} b_n(p)\sin\frac{n\pi y}{2h}, \\
\frac{\partial\bar{u}^r}{\partial x}(0, y, p) &= -\left(\frac{k^2-2}{k^2}\right)\sum_{n=1,3,5,\ldots}^{\infty}\frac{n\pi}{2h}b_n(p)\cos\frac{n\pi y}{2h}, \\
\frac{\partial\bar{v}^r}{\partial x}(0, y, p) &= \sum_{n=1,3,5,\ldots}^{\infty}\frac{n\pi}{2h}a_n(p)\sin\frac{n\pi y}{2h},
\end{aligned} \tag{3.6}$$

†Since the plate-face stresses $\sigma'_y$, $\tau'_{xy}$ of (3.5) decay as $x \to \infty$, Problem B2 is amenable to a finite-element treatment. Such a treatment is outlined in Appendix 1, [4].

for $0 \leq y \leq h$, $Re\, p > 0$. Here, the $r$ atop quantities denotes their regular nature; the $\bar{u}^c(p)$ is once again the transformed supplemental corner displacement.

Contingent upon the validity of the thesis that the forms in (3.3) will describe the singular contributions to the edge unknowns in their entirety, the terms in (3.6) will in fact be regular and consequently the $a_n(p)$ and $b_n(p)$ there will comply with the large-$n$ order condition of (2.7). Moreover, such regularity then guarantees the validity of term-by-term differentiation of $\bar{u}^r$, $\bar{v}^r$ in (3.6). Thus, on adjoining (3.3) to (3.6) and the terms $(\partial\bar{u}^r/\partial y)$, $(\partial\bar{v}^r/\partial y)$ obtained therefrom, the postulation of the edge unknowns for Problem B is concluded.†

We are now in a position to open up the boundedness condition and evaluate the edge unknowns for small $p$. To achieve this we closely follow the method established for Problem A in Section 2. The forms for the edge unknowns for Problem B are substituted into the boundedness condition (1.16) and the expanded (1.16) integrated. This results in an infinite system of linear equations for the unknowns $\bar{u}^c(p)$, $a_n(p)$ and $b_n(p)$ which is similar to (2.9), the only differences encountered being contained in an $F_B(s, p)$ term. The system has one equation for each $s_j(p)$ satisfying (1.15). For $p$ small (as argued in Section 2) $s_j(p) \to s_j$ ($j = 1, 2, \ldots$), the roots of $r(s)$ in (2.11), and $\bar{u}^c(p)$, $a_n(p)$ and $b_n(p)$ obey the order requirements (2.13). In view of these order requirements, the boundedness condition for Problem B reduces to,

$$\sum_{n=1,3,5,\ldots}^{\infty} (-)^{(n+1)/2} \frac{n^2 \hat{s}}{\hat{s}_n^{\,4}} \left[\frac{n\pi}{2}\hat{a}_n + \hat{s}\hat{b}_n\right] = \frac{2}{\pi^2 \cos \hat{s}} [(1 + \hat{s} \tan \hat{s}) si(\hat{s}) + \sin \hat{s}], \tag{3.7}$$

for $p \to 0$, $\hat{s} = \hat{s}_j$ ($j = 1, 2, \ldots$); wherein all quantities have been rendered dimensionless by the introduction of $\hat{a}_n = pa_n(p)\sigma_B h$, $\hat{b}_n = pb_n(p)/\sigma_B h$, $\hat{s} = sh$, $\hat{s}_n^{\,2} = s_n^{\,2} h^2$, with $\sin 2\hat{s}_j + 2\hat{s}_j = 0$. Here $si(s)$ is the sine integral.‡

As in [1, 2], we now employ the *method of reduction*§ to evaluate $\hat{a}_n$, $\hat{b}_n$. Results for the first ten $\hat{a}_n$ and $\hat{b}_n$ found using twenty-four roots $\hat{s}_j$, are displayed in the following table.

Table 1

Fourier coefficient values

| $n$ | $\hat{a}_n$ | $\hat{b}_n$ |
|---|---|---|
| 1 | −0·1129 | 0·2891 |
| 3 | −0·0242 | 0·0036 |
| 5 | 0·0077 | −0·0026 |
| 7 | −0·0030 | 0·0011 |
| 9 | 0·0014 | −0·0005 |
| 11 | −0·0007 | 0·0003 |
| 13 | 0·0004 | −0·0001 |
| 15 | −0·0003 | 0·0001 |
| 17 | 0·0002 | −0·0001 |
| 19 | −0·0001 | 0·0000 |

The numerical decay of the $\hat{a}_n$, $\hat{b}_n$ values in the table is faster than $1/n^2$ for $n > 3$, in agreement with our large-$n$ order condition (2.7). Such numerical decay supports our thesis that in the

†Details of the forms for the edge unknowns for Problem B and of the ensuing opening up of the boundedness condition utilizing the forms are given in [4].

‡See Abramowitz and Stegun[7], p. 232.

§That is, solving the finite $2N \times 2N$ system of linear equations associated with the first $N$ roots $\hat{s}_j$, then increasing the size of the finite system solved until stable estimates of the desired number of $\hat{a}_n$ and $\hat{b}_n$ have been found.

long-time near-field the singular nature of Problem B is the same as for the corresponding static problem.

Using the $\hat{a}_n$, $\hat{b}_n$ values of the table, the edge displacements associated with Problem B2 can be evaluated, enabling comparison with the finite-element analysis given in Appendix 1, [4]. Such a comparison shows agreement to within 1 per cent.

The edge displacements associated with Problem B, however, cannot be evaluated at this juncture since presently $\bar{u}^c(p)$ is an unknown. To ascertain $\bar{u}^c(p)$ we proceed as in Problem A and consider the boundedness condition for the case of $s$ and $p$ tending to zero together. This limiting process defines $s_0$ as given in (2.15) and the boundedness condition associated with $s_0$ then furnishes the necessary additional equation for the determination of $\bar{u}^c(p)$, namely

$$\bar{u}^c(p) = \frac{\hat{\sigma}_B h}{p}\left[1 + \sum_{n=1,3,5,\ldots}^{\infty} (-)^{(n-1)/2} \frac{2}{n\pi} a_n\right] \quad \text{as} \quad p \to 0. \tag{3.8}$$

Substituting our numerical values for $\hat{a}_n$ into (3.8) then gives $u^c(p) = \hat{\sigma}_B \hat{u}^c h/p$ with $\hat{u}^c = 0{\cdot}935$.

We have now completed the determination of the edge unknowns for Problem B. For the purposes of exhibiting these results we remove the $\pi\hat{\sigma}_B/4pk_p$ term, carry out the simple inversion of the remaining terms and then define the *static edge displacements* (i.e. time independent displacements) and their *gradients* as follows:

$$\hat{u} = \left[u(0, y, t) - \frac{\pi}{4}\hat{\sigma}_B c_p t\right] \Big/ \hat{\sigma}_B h = \bar{u}^c - \ln\frac{y}{n} + \sum_{n=1,3,5,\ldots}^{\infty} \hat{a}_n \cos\frac{n\pi y}{2h},$$

$$\hat{v} = v(0, y, t)/\hat{\sigma}_B h = \frac{-\pi}{2k^2}\operatorname{sgn}(y) + \sum_{n=1,3,5,\ldots}^{\infty} \hat{b}_n \sin\frac{n\pi y}{2h}, \tag{3.9}$$

$$\frac{\partial \hat{u}}{\partial y} = -\left[\frac{h}{y} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi}{2}\hat{a}_n \sin\frac{n\pi y}{2h}\right], \quad \frac{\partial \hat{v}}{\partial y} = \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\pi}{2}\hat{b}_n \cos\frac{n\pi y}{2h},$$

for $0 < y \le h$, with $(\partial\hat{v}/\partial y)$ having a symmetric delta at $y = 0$. Using our numerical values of $\hat{a}_n$, $\hat{b}_n$ and setting $k^2 = 7/2$ affords a means of calculating $\hat{u}$, $\hat{v}$, $(\partial\hat{u}/\partial y)$ and $(\partial\hat{v}/\partial y)$ of (3.9). The results of this calculation are plotted in Fig. 6.

In proceeding to the small-$p$ formal solution for Problem B, and thus to the long-time solution, we substitute the now determined edge unknowns into (1.9) by means of equations (1.10), (1.11), (1.13) and (1.14). Thereafter the inversion process is undertaken for the pertinent three ranges of $s$ in exactly the same manner as in Section 2. For the *near-field* this process produces the edge displacements and their gradients. In the *far-field*, the process gives

$$u(x, y, t) \sim \frac{\hat{\sigma}_B\pi}{4}[c_p t - x]U(c_p t - x), \quad v(x, y, t) \sim \frac{\hat{\sigma}_B\pi}{4} yU(c_p t - x),$$
$$\frac{\partial u}{\partial x}(x, y, t) \sim -\left(\frac{k^2}{k^2-2}\right)\frac{\partial v}{\partial y}(x, y, t) \sim -\frac{\hat{\sigma}_B\pi}{4} U(c_p t - x), \tag{3.10}$$

for $t \to \infty$ on $D$, as the *first approximation* with

$$\frac{\partial u}{\partial x}(x, y, t) \sim -\left(\frac{k^2}{k^2-2}\right)\frac{\partial v}{\partial y}(x, y, t) \sim -\frac{\hat{\sigma}_B\pi}{4}\left[\int_0^{\eta} Ai(-\eta')\,d\eta' + \frac{1}{3}\right] \tag{3.11}$$

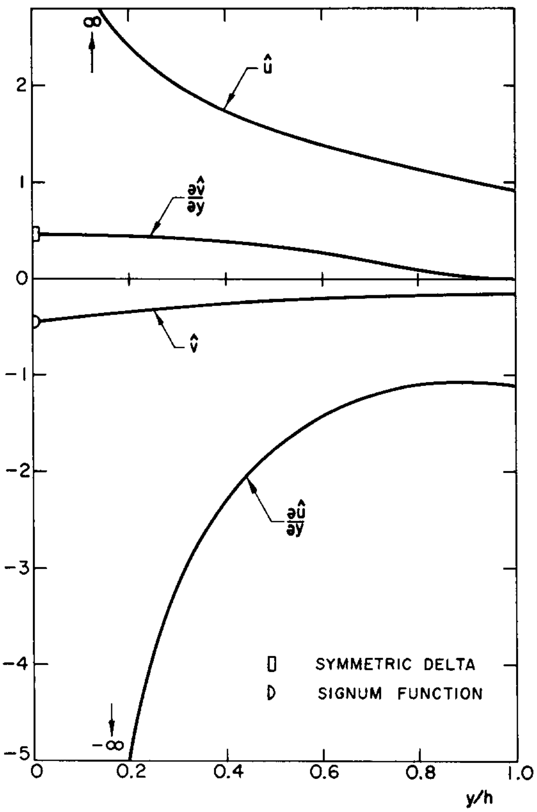


Fig. 6. Static edge displacements and derivatives thereof for Problem B.

for $x, t \to \infty$, as the *second approximation.* Comparison of (2.21), (2.22) with (3.10) and (2.25), (2.27) with (3.11) demonstrates that the long-time far-field approximations for Problems A and B are the same if equal normal forces act on the edge, $x = 0$, in both problems—that is, if $\sigma_A = \sigma_B/2h$ or equivalently $\hat{\sigma}_A = \pi\hat{\sigma}_B/4$.

## REFERENCES

1. J. Miklowitz, Analysis of elastic waveguides involving an edge, in *Wave Propagation in Solids.* ASME Publications, N.Y. (1969).
2. J. Benthem, A Laplace transform method for the solution of semi-infinite and finite strip problems in stress analysis, *Quart. J. Mech. Appl. Math.* **16**, 413 (1963).
3. J. Miklowitz, On solving elastic waveguide problems involving non-mixed edge conditions, *ONR Technical Report* No. 5, Contract Nonr-220(57), California Institute of Technology (1967).
4. G. B. Sinclair, Ph.D. Dissertation, California Institute of Technology (1972).
5. C. I. Robbins and R. C. T. Smith, A table of the roots of $\sin z = -z$. *Phil. Mag.* **39** (7), 1004 (1948).
6. R. Skalak, Longitudinal impact of a semi-infinite circular elastic bar, *J. Appl. Mech.* **24**, 59 (1957).
7. *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun. National Bureau of Standards (1964).

## APPENDIX

*Derivation of the singular parts of the edge unknowns cited in* (3.3)

We indicate here how one derives the additional information for the familiar Flamant problem which is required to establish the singular parts of the edge unknowns. In essence the derivation involves solving the Flamant problem (Problem B1) as the limit $\Delta \to 0$ of the sequence of *kernel problems* indicated by $\delta(y;\Delta)$.

A kernel problem will thus constitute a second boundary-value problem in elastostatics for the half-space $(0 \le x < \infty,\ -\infty < y < \infty)$ with: $\sigma_x(0,y;\Delta) = -\sigma_B\delta(y;\Delta)$, $\tau_{xy}(0,y;\Delta) = 0$ for $-\infty < y < \infty$, $\Delta > 0$; and $\sigma_x(x,y;\Delta)$, $\sigma_y(x,y;\Delta)$, $\tau_{xy}(x,y;\Delta)$ all $o(1)$ as $r \to \infty$, $r = \sqrt{x^2+y^2}$. These problems are tractable to Fourier transformation on $y$ as a means of deriving their solutions. In what follows we confine our attention to the results of such a derivation.

Consider the displacement field

$$
\begin{aligned}
u(x,y;\Delta) &= \frac{\hat{\sigma}_B h}{2\Delta k^2}\left[x\left\{\arctan\left(\frac{y-\Delta}{x}\right) - \arctan\left(\frac{y+\Delta}{x}\right)\right\}\right. \\
&\quad \left. + k^2\{(y-\Delta)\ln\sqrt{x^2+(y-\Delta)^2} - (y+\Delta)\ln\sqrt{x^2+(y+\Delta)^2}\}\right], \\
v(x,y;\Delta) &= \frac{\hat{\sigma}_B h}{2\Delta k^2}\left[(y-\Delta)\arctan\left(\frac{y-\Delta}{x}\right) - (y+\Delta)\arctan\left(\frac{y+\Delta}{x}\right)\right. \\
&\quad \left. + k^2 x\{\ln\sqrt{x^2+(y+\Delta)^2} - \ln\sqrt{x^2+(y-\Delta)^2}\}\right],
\end{aligned}
\tag{1}
$$

for $x \ge 0$, $-\infty < y < \infty$, and wherein the arctangent ranges from $-\pi/2$ to $\pi/2$. Differentiation of (1) and substitution of the terms found thereon into the stress–displacement relations (*viz.* (1.2) without the $t$-dependence) produces a stress field which satisfies the boundary conditions and the conditions at infinity for a kernel problem. A second differentiation and substitution into the time-independent counterpart of (1.1) reveals that $u(x, y;\Delta)$, $v(x, y;\Delta)$ of (1) satisfy the displacement equations of equilibrium. Consequently $u(x,y;\Delta)$, $v(x,y;\Delta)$ of (1) comprise the appropriate values for a kernel problem.

Now for $x = 0$, (1) yields

$$
\begin{aligned}
u(0,y;\Delta) &= \frac{\hat{\sigma}_B h}{2\Delta}[(y-\Delta)\ln|y-\Delta| - (y+\Delta)\ln|y+\Delta|], \\
\frac{\partial u}{\partial y}(0,y;\Delta) &= -\frac{\partial v}{\partial x}(0,y;\Delta) = \frac{\hat{\sigma}_B h}{2\Delta}\ln\left|\frac{y-\Delta}{y+\Delta}\right|, \\
v(0,y;\Delta) &= -\hat{\sigma}_B\frac{\pi h}{2k^2}\,\mathrm{sgn}\,(y;\Delta), \\
\frac{\partial v}{\partial y}(0,y;\Delta) &= \frac{\partial u}{\partial x}(0,y;\Delta) = -\hat{\sigma}_B\frac{\pi h}{k^2}\,\delta(y;\Delta),
\end{aligned}
\tag{2}
$$

where $\mathrm{sgn}\,(y;\Delta) = (1 - y/\Delta)U(y-\Delta) + (1 + y/\Delta)U(y+\Delta) - 1$. Proceeding to the limit $\Delta \to 0$ in (2) then gives the elastostatic analogue of (3.3).